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Convergence of Abstract Splines

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Shekhtman (*J. Approx. Theory* **30**(1980), 237–246) gives a sufficient condition for the convergence of abstract splines. We show that his condition is not necessary and give a related condition which is both necessary and sufficient. In the process, we also give a necessary and sufficient condition for a sequence of abstract spline projectors to be bounded.

Shekhtman [3] gives a sufficient condition for the convergence of abstract splines. We give a related condition which is both necessary and sufficient. In the process, we also give a necessary and sufficient condition for a sequence of abstract spline projectors to be bounded.

It seems most convenient to discuss the abstract spline (as introduced by Atteia [1]) in the following way. Let X be a Hilbert space, and let Λ be a set of continuous linear functionals on X. From the possibly many elements of X which agree with a given $x \in X$ on Λ, i.e., from the flat

$$x + \ker \Lambda,$$

we attempt to select a particular one by the prescription that it should minimize $\|Ty\|$ over y in $x + \ker \Lambda$. Here,

$$\ker \Lambda := \Lambda_\perp := \bigcap_{\lambda \in \Lambda} \ker \lambda$$

and T is a given bounded linear map on X to some Hilbert space Z. Any solution to this minimization problem is an abstract spline, or, more precisely, a (T, Λ)-*spline interpolant* to the x in question.

We now make some preliminary remarks regarding these interpolants.

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(i) The minimization problem and its solution(s) depend only on $\ker \Lambda$, hence do not change if we replace Λ by its closed linear hull, i.e., by $(\Lambda_\perp)^\perp = (\Lambda\perp)\perp$. We therefore assume from now on that

Λ is a closed linear subspace of X^* $(=X)$.

(ii) In order to guarantee existence and uniqueness of the (T, Λ)-spline interpolant for every x in X (and other things besides), Shekhtman [3] makes the assumption that

$$\begin{aligned} &\ker T \cap \ker \Lambda = \{0\} \\ &\operatorname{ran} T \quad \text{is closed} \\ &\dim \ker T < \infty \end{aligned} \tag{1}$$

Although the finite dimensionality of $\ker T$ is required for some of Shekhtman's arguments, it turns out to be unnecessarily restrictive. Instead, I assume that

$$\begin{aligned} &\operatorname{incl}(\ker T, \ker \Lambda) < 1 \\ &\operatorname{ran} T \text{ is closed} \end{aligned} \tag{1'}$$

Here, the *inclination* between two subspaces A and B is, by definition, the cosine of the smallest angle between them, i.e., the number

$$\operatorname{incl}(A, B) := \sup_{a \in A, b \in B} \frac{(a, b)}{\|a\| \, \|b\|} = \|P_A|_B\| = \|P_B|_A\|, \tag{2}$$

with P_A, P_B the orthogonal projector onto A and B, respectively. In particular, it is easy to see that $\ker T \cap \ker \Lambda = \{0\}$ is equivalent to $\operatorname{incl}(\ker T, \ker \Lambda) < 1$ in case $\ker T$ (or $\ker \Lambda$) is finite dimensional.

Assumption (1′) is motivated by the following lemma whose proof I give here for completeness.

LEMMA 1. *Assume that* $\operatorname{ran} T$ *is closed. Then there exists one and only one (T, Λ)-spline interpolant for each x in X if and only if*

$$\operatorname{incl}(\ker T, \ker \Lambda) < 1. \tag{3}$$

Proof. We have

$$\inf_{y \in x + \ker \Lambda} \|Ty\| = \inf_{z \in \ker \Lambda} \|Tx - Tz\| = \operatorname{dist}(Tx, T[\ker \Lambda]),$$

hence, y is a (T, Λ)-spline interpolant to x iff $x - y \in \ker \Lambda$ and Ty is the error in *the* best approximation to Tx from $T[\ker A]$. On the other hand, $\operatorname{ran} T$ is closed by assumption, hence contains the closure of $T[\ker A]$, therefore every Tx in $\operatorname{ran} T$ has a best approximation from $T[\ker A]$ iff $T[\ker A]$ is closed. (Here I use the fact that $T[\ker A]$ is a linear subspace of a Hilbert space.) This shows that *every x in X has a (T, A)-spline interpolant iff $T[\ker A]$ is closed.*

Further, since the difference between two (T, A)-spline interpolants to x necessarily lies in $\ker A$ as well as in $\ker T$ (since T must map them to the same point, viz. the error in the best approximation to Tx), *there is at most one (T, A)-spline interpolant for a given x iff* $\ker T \cap \ker A = \{0\}$.

This shows existence and uniqueness of the interpolant to be equivalent to

$$\ker T + \ker A (= T^{-1}[T[\ker A]]) \text{ is closed and } \ker T \cap \ker A = \{0\}, \tag{4}$$

provided we can prove that (4) implies that $T[\ker A]$ is closed. For this, if (4) holds, then X is the topological direct sum of $\ker T$ and $\ker A + (\ker T + \ker A)\perp$. This latter subspace is mapped 1–1 onto $\operatorname{ran} T$ and so, $\operatorname{ran} T$ being closed by assumption, this mapping is open. In particular, $T[\ker A]$ must then be closed.

This leaves the task of showing that (4) and (3) are equivalent. For this, observe (else, see, e.g., [4; p. 243, Problem 3]) that (4) is equivalent to

$$\inf\{\|x - y\| : x \in \ker T, y \in \ker A, \|x\| = \|y\| = 1\} > 0$$

and, since X is a Hilbert space hence

$$\|x\| = \|y\| = 1 \qquad \text{implies} \qquad \|x - y\|^2 = 2 - 2(x, y),$$

this is obviously equivalent to (3). ∎

(iii) In conclusion, assumption (1′) ensures that the minimization problem has exactly one solution for given x. We shall denote it by

$$px.$$

It is obvious that *the map p so defined is a linear projector on X*, with

$$\ker p = \ker A.$$

Further, p is a *bounded* linear projector. It will be important later on to know, more precisely, that $\|p\|$ can be bounded above and below in terms of

$$c := \operatorname{incl}(\ker T, \ker A)$$

as the following proposition shows.

PROPOSITION 1. *Let* $s := \sin(\ker T, \ker \Lambda) := \sqrt{1-c^2}$. *Then*

$$1/s \leqslant \|p\| \leqslant 1 + \|(T|_{\operatorname{ran} Q})^{-1}\|/s$$

with

$$Q := 1 - P_{\ker T}$$

the orthoprojector onto $(\ker T)\perp$.

Proof. For the lower bound, let $P := P_\Lambda$ so that $\ker P = \ker \Lambda = \ker p$. Since $px = x$ for x in $\ker T$, we have

$$\|x\| = \|px\| \leqslant \|p\| \operatorname{dist}(x, \ker p), \qquad \text{for all} \quad x \in \ker T,$$

while $\operatorname{dist}(x, \ker p) = \operatorname{dist}(x, \ker P) = \|Px\|$. Consequently,

$$\|p\| \geqslant \sup_{x \in \ker T} \frac{\|x\|}{\|Px\|},$$

while

$$\begin{aligned} \inf_{x \in \ker T} \left(\frac{\|Px\|}{\|x\|}\right)^2 &= 1 - \sup_{x \in \ker T} \frac{\|(1-P)x\|^2}{\|x\|^2} \\ &= 1 - \|(1-P)|_{\ker T}\|^2 = 1 - c^2 \end{aligned}$$

using (2) and the fact that $1 - P = P_{\ker \Lambda}$.

For the upper bound, recall from Golomb [2, (3.8)] (or else verify directly) that

$$p = 1 - T_0^{-1}(P_{T[\ker \Lambda]})T \tag{5}$$

with $T_0 := T|_{\ker \Lambda}$. Consequently,

$$\|p\| \leqslant 1 + \|T_0^{-1}\| \, \|T\|,$$

and we calculate $\|T_0^{-1}\|$ as

$$\|T_0^{-1}\| = \sup_{x \in \ker \Lambda} \|x\|/\|Tx\|.$$

But, since $Tx = TQx$ (using the orthoprojector Q onto $(\ker T)\perp$ introduced earlier), we have

$$\|x\|/\|Tx\| = \frac{\|x\|}{\|Qx\|} \frac{\|Qx\|}{\|TQx\|}$$

hence

$$\|T_0^{-1}\| \leqslant \sup_{x \in \ker \Lambda} (\|x\|/\|Qx\|)\, \|(T|_{\operatorname{ran} Q})^{-1}\|$$

while, as before,

$$\inf_{x \in \ker \Lambda} \frac{\|Qx\|^2}{\|x\|^2} = 1 - \sup_{x \in \ker \Lambda} \frac{\|(1-Q)x\|^2}{\|x\|^2} = 1 - \|(1-Q)|_{\ker \Lambda}\|^2 = 1 - c^2$$

by (2) and since $1 - Q = P_{\ker T}$. ∎

We record for later use the following result obtained during the proof of the upper bound:

$$\|(T|_{\ker \Lambda})^{-1}\| \leqslant \|(T|_{\operatorname{ran} Q})^{-1}\|/s. \tag{6}$$

Further, the proof of the lower bound provides the following convenient criterion (as well as the criterion obtained from it by interchanging $\ker T$ and $\ker \Lambda$ throughout).

COROLLARY. $\operatorname{incl}(\ker T, \ker \Lambda) < 1$ *iff there exists a* bounded *linear projector P with* $\ker P = \ker \Lambda$ *and* $\operatorname{ran} P \supseteq \ker T$.

Proof. The argument for the first inequality in Proposition 1 uses only that p is a bounded linear projector with kernel equal to $\ker \Lambda$ and range containing $\ker T$, hence proves that

$$1/\sin(\ker T, \ker \Lambda) \leqslant \inf\{\|P\|: P \text{ l.proj.}, \ker P = \ker \Lambda, \operatorname{ran} P \supseteq \ker T\}.$$

and so shows, in particular, that $\operatorname{incl}(\ker T, \ker \Lambda) < 1$ in case such a projector exists. On the other hand, Lemma 1, for example, in conjunction with Proposition 1 shows the existence of such a projector (viz. p) in case $\operatorname{incl}(\ker T, \ker \Lambda) < 1$. ∎

We now come to the point of this note. Let (Λ_n) be a given sequence of closed subspaces of $X^* = X$ satisfying

$$\operatorname{incl}(\ker T, \ker \Lambda_n) < 1, \text{ all } n.$$

Then Shekhtman is concerned with the question of when the corresponding sequence (p_n) of spline projectors converges pointwise, or strongly, to 1. In this connection, the following well-known lemma is a consequence of the uniform boundedness principle and Lebesgue's Inequality

$$\|x - p_n x\| \leqslant \|1 - p_n\| \operatorname{dist}(x, \operatorname{ran} p_n).$$

LEMMA 2. *$p_n \longrightarrow^s 1$ iff (p_n) is bounded and $\underline{\lim}_{n\to\infty}$ ran $p_n = x$.*

Here, we use the abbreviation

$$\underline{\lim}\, A_n := \{\lim a_n \colon a_n \in A_n, \text{ all } n\}$$

with $\lim a_n$ taken in the norm on X.

Unfortunately, the spline interpolation projector is given in terms of T and (A_n) and the character of ran p_n is, in general, not known a priori. It is therefore important to give conditions for the convergence of p_n in terms of T and (A_n).

THEOREM 1 (Shekhtman [3]). *If* $\dim\ker T < \infty$, *then* $\underline{\lim}\, A_n = X$ *implies* $p_n \longrightarrow^s 1$.

The major part of the proof is spent in proving

LEMMA 3. *If* $\dim\ker T < \infty$ *and* $\underline{\lim}\, A_n = X$, *then* (p_n) *is bounded.*

I want to give a different proof of this lemma. By Proposition 1, (p_n) in bounded iff

$$\sup \operatorname{incl}(\ker T, \ker A_n) < 1. \tag{7}$$

This latter condition is trivially satisfied in case (A_n) is increasing (the only situation considered, e.g., in Golomb [2]) since then $\operatorname{incl}(\ker T, \ker A_n)$ is decreasing as n increases. Condition (7) is also satisfied in case $\underline{\lim}\, A_n \supseteq \ker T$ (and $\dim\ker T < \infty$). For, if (7) were violated, there would exist, using the fact that $\dim\ker T < \infty$, an x in $\ker T$ and y_n in $\ker A_n$, all n, so that

$$\overline{\lim} \frac{(x, y_n)}{\|x\|\,\|y_n\|} = 1.$$

But then, for all z_n in A_n,

$$\overline{\lim} \frac{\|x - z_n\|}{\|x\|} \geqslant \overline{\lim} \frac{|(x - z_n, y_n)|}{\|x\|\,\|y_n\|} = \overline{\lim} \frac{|(x, y_n)|}{\|x\|\,\|y_n\|} = 1,$$

showing that x would not be in $\underline{\lim}\, A_n$. In particular, Lemma 3 follows.

Shekhtman finishes the proof of Theorem 1 with the following nice observation: Since (p_n) is bounded, so is (p_n^*), and, since $\operatorname{ran} p_n^* = A_n$ while $\underline{\lim}\, A_n = X$, by assumption, it follows that $p_n^* \longrightarrow^s 1$. Consequently, $p_n \longrightarrow^w 1$. But then $Tp_n \longrightarrow^w T$, therefore $\|Tx\| \leqslant \underline{\lim}\, \|Tp_n x\|$, while also $\|Tp_n x\| \leqslant \|Tx\|$. Therefore $\|Tp_n x\| \to \|Tx\|$, and so $Tp_n \longrightarrow^s T$. It follows that

$$Qp_n = (T|_{\operatorname{ran} Q})^{-1} Tp_n \xrightarrow{s} (T|_{\operatorname{ran} Q})^{-1} T = Q$$

while, by the finite dimensionality of $\ker T = \operatorname{ran}(1-Q)$, $p_n \longrightarrow^w 1$ implies $(1-Q)p_n \longrightarrow^s 1-Q$. ∎

Since $\operatorname{ran} p_n^* = \Lambda_n$ while $\|p_n^*\| = \|p_n\|$, Shekhtman's argument shows that, for the particular sequence (p_n) of spline projectors,

$$p_n^* \xrightarrow{s} 1 \qquad \text{implies} \qquad p_n \xrightarrow{s} 1$$

(at least in case $\dim \ker T < \infty$). Such an implication does not hold for general sequences of linear projectors, so that the converse of Theorem 1, if true, would again have to be proved using some special properties of the spline projectors. As it turns out, though, the converse does not hold even for spline projectors, as the following simple example shows.

Example. Take $X = Z = l_2$, $T = Q$, $1 - Q = P_{\operatorname{span}\{e_1\}}$, with $e_j := (\delta_{ij})_{i=1}^\infty$ and

$$\Lambda_n = \operatorname{span}\{e_2, ..., e_{n-1}, e_n + e_1\}.$$

Then $p_n x = \sum_{j<n} x(j) e_j + x(n) e_1$ which converges in norm to x since $\lim x(n) = 0$. In other words, $p_n \longrightarrow^s 1$. On the other hand,

$$\operatorname{dist}(e_1, \Lambda_n) = \operatorname{dist}(e_1, \operatorname{span}\{e_1 + e_n\}) = 1/\sqrt{2},$$

i.e., $e_1 \notin \underline{\lim}\, \Lambda_n$.

In this example, $\underline{\lim}\, \Lambda_n = \operatorname{span}\{e_2, e_3, ...\} = (\ker T)\bot$, hence

$$\underline{\lim}\, \Lambda_n \supseteq (\ker T)\bot. \tag{8}$$

I will show below that condition (8) is necessary for $p_n \longrightarrow^s 1$. The example then also shows that $\underline{\lim}\, \Lambda_n$ need not contain anything else. First, though, I want to settle under what circumstances the converse of Theorem 1 holds.

Proposition 2. *Suppose that* $\dim \ker T < \infty$ *and* $p_n \longrightarrow^s 1$. *Then* $\underline{\lim}\, \Lambda_n = X$ *if and only if there exists a linear projector R with* $\operatorname{ran} R = \ker T$ *which is the uniform limit of a sequence (R_n) of linear projectors with* $\operatorname{ran} R_n = \ker T$ *and* $\operatorname{ran} R_n^* \subseteq \Lambda_n$, *all $n \geqslant n_0$.*

Proof. Since $\dim \ker T < \infty$, any bounded linear projector R on X with range $\ker T$ can be written

$$R = \sum_{i=1}^{r} x_i \otimes \lambda_i$$

for some basis $(x_i)_1^r$ of $\ker T$ and some dual set $(\lambda_i)_1^r$ of linear functionals.

But, if now $\underline{\lim} \Lambda_n = X$, then we can find sequences $(\lambda_i^{(n)})$ with $\lambda_i^{(n)} \in \Lambda_n$, all n, and $\|\lambda_i - \lambda_i^{(n)}\| \to 0$, $i = 1,..., r$. Since $\lambda_i x_j = \delta_{ij}$, all i, j, it is then also possible for all large enough n to find a basis $(x_i^{(n)})$ for $\ker T$ with $\lambda_i^{(n)} x_j^{(n)} = \delta_{ij}$ and then, necessarily, also $\|x_i - x_i^{(n)}\| \longrightarrow_{n\to\infty} 0$. But then

$$R_n := \sum_{i=1}^{r} x_i^{(n)} \otimes \lambda_i^{(n)}$$

converges in norm to R.

For the converse, if R_n converges in norm to R, then the sequence (S_n) given by

$$S_n := R_n^* R_n + T^*T$$

converges in norm to

$$S := R^*R + T^*T.$$

The linear map S is selfadjoint, bounded, and is bounded below. Explicitly,

$$(Sx, x) = \|Rx\|^2 + \|Tx\|^2$$

while $TRx = 0$, hence

$$\|Tx\|^2 = \|T(1-R)x\|^2 \in \{\|(T|_{\operatorname{ran}(1-R)})^{-1}\|, \|T\|\}^2 \|(1-R)x\|^2.$$

This shows that

$$(Sx, x) \in \{\min\{1, \|(T|_{\operatorname{ran}(1-R)})^{-1}\|\}, \max\{1, \|T\|\}\}^2(\|Rx\|^2 + \|(1-R)x\|^2)$$

while

$$\|Rx\|^2 + \|(1-R)x\|^2 \in \{\tfrac{1}{2}, 1 + 2\|R\| \|1-R\|\} \|x\|^2.$$

We conclude that the bilinear form

$$(x, y)_S := (Sx, y)$$

is an equivalent inner product on X and S is, therefore, in particular invertible. Since $S_n \to S$ in norm, it follows that also S_n^{-1} exists for n sufficiently large and converges in norm to S^{-1}.

We now conclude from $p_n \xrightarrow{s} 1$ that also $S_n p_n S_n^{-1} \xrightarrow{s} 1$. In particular, for $x \in X$, setting $z_n := S_n^{-1} x$, we get

$$x \underset{\infty \leftarrow n}{\longleftarrow} S_n p_n z_n = R_n^* R_n p_n z_n + T^*T p_n z_n.$$

By construction, $\operatorname{ran} R_n^* \subseteq \Lambda_n$, while $T^*Tp_n[X] \subseteq \Lambda_n$ due to the fact that (e.g., by (5)) $Tp_n = (1 - P_{T[\ker \Lambda_n]})T$, hence $Tp_n[X] \subseteq T[\ker \Lambda_n]\perp$ and so $T^*Tp_n[X] \subseteq (\ker \Lambda_n)\perp = \Lambda_n$. But this shows that $x \in \underline{\lim}\, \Lambda_n$. ∎

The argument for the converse does not use the finite dimensionality of ker T and therefore shows, carried out with $R_n = 1 - Q$, all n (recall that $Q = P_{(\ker T)\perp}$), that, for all $x \in X$,

$$(1 - Q)\, p_n z_n + T^*Tp_n z_n \xrightarrow[n\to\infty]{} x.$$

But, since ran $T^* \subseteq (\ker T)\perp = \operatorname{ran} Q$, this implies that $T^*Tp_n z_n \to Qx$ and so shows that $(\ker T)\perp = \operatorname{ran} Q \subseteq \underline{\lim}\, T^*Tp_n[X] \subseteq \underline{\lim}\, \Lambda_n$. This proves

COROLLARY. *If* $p_n \xrightarrow{s} 1$, *then* $(\ker T)\perp \subseteq \underline{\lim}\, \Lambda_n$.

THEOREM 2. $p_n \xrightarrow{s} 1$ *iff* $\sup \operatorname{incl}(\ker T, \ker \Lambda_n) < 1$ *and* $(\ker T)\perp \subseteq \underline{\lim}\, \Lambda_n$.

Proof. Proposition 1 and the corollary to Proposition 2 show (with Lemma 2) that the stated conditions are necessary for $p_n \xrightarrow{s} 1$. In order to show the sufficiency of these conditions, we need, by Proposition 1 and Lemma 2, only prove the following

PROPOSITION 3. *If* (p_n) *is bounded and* $(\ker T)\perp \subseteq \underline{\lim}\, \Lambda_n$, *then* $p_n \xrightarrow{s} 1$.

Proof. Since $\ker T \subseteq \operatorname{ran} p_n$ and (p_n) is bounded by assumption, we are done once we show that $(\ker T)\perp \subseteq \underline{\lim}\operatorname{ran} p_n$. For this, let $z \in (\ker T)\perp = \operatorname{ran} Q$, and consider $y := T^*Tz$, also in ran Q. By assumption, $y = \lim y_n$, with $y_n \in \Lambda_n$, all n. Consequently,

$$\lim Qy_n = T^*Tz \qquad \text{and} \qquad \lim (1 - Q)\, y_n = 0. \tag{9}$$

Now consider the bounded and boundedly invertible linear map

$$S := 1 - Q + T^*T$$

on X introduced earlier for the proof of the corollary to Proposition 2. Note that ker T and $(\ker T)\perp = \operatorname{ran} Q$ are both invariant under S, and $S = 1$ on ker T. Hence we can write y_n as

$$y_n = (1 - Q)\, y_n + T^*Tz_n$$

for some $z_n \in \operatorname{ran} Q$ and, since $y_n \to y \in \operatorname{ran} Q$, we have $T^*Tz_n \to T^*Tz$, thus $z_n \to z$. Hence we need only prove that $z_n - p_n z_n \to 0$. For this, we have from (5) that

$$z_n - p_n z_n = (T|_{\ker \Lambda_n})^{-1}\, P_{T[\ker \Lambda_n]}\, Tz_n$$

while, by (6), Proposition 1 and the boundedness of (p_n),

$$\sup_n \|(T|_{\ker \Lambda_n})^{-1}\| < \infty. \tag{10}$$

Thus, we need only show that $\|P_{T[\ker \Lambda_n]} Tz_n\| \to 0$. For this, note that

$$\|P_{T[\ker \Lambda_n]} Tz_n\| = \sup_{x \in \ker \Lambda_n} \frac{|(Tx, Tz_n)|}{\|Tx\|}$$

while, for all $x \in \ker \Lambda_n$,

$$0 = (x, y_n) = (x, (1-Q)\, y_n) + (x, T^*Tz_n),$$

hence

$$|(Tx, Tz_n)| \leqslant \|x\|\, \|(1-Q)\, y_n\|.$$

Therefore

$$\begin{aligned}\|P_{T[\ker \Lambda_n]} Tz_n\| &\leqslant \sup_{x \in \ker \Lambda_n} \frac{\|x\|\, \|(1-Q)\, y_n\|}{\|Tx\|}\\ &= \|(T|_{\ker \Lambda_n})^{-1}\|\, \|(1-Q)\, y_n\| \xrightarrow[n\to\infty]{} 0,\end{aligned}$$

the last by (9) and (10). ∎

Remark. In effect, the proof of Propositions 2 and 3 relies on the fact that T^*T maps ran $p_n \cap$ ran Q 1–1 onto ran $p_n^* \cap$ ran Q.

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